HEAT CONDUCTION WITH A TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY COEFFICIENT

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A variational method is employed to solve stationary and nonstationary heat conduction problems when the thermal conductivity coefficient is temperature-dependent and the heat generation function of the medium is arbitrary.

The Variational Formulation

1. In situations where temperature drops are large and an accurate temperature distribution is to be determined, the temperature-dependence of the thermophysical parameters must be taken into account. This is the case, for example, in nuclear reactors when calculations are made of the maximum possible power [1]. Unfortunately, if the coefficient of thermal conductivity cannot be considered constant, the mathematical problem becomes very involved and leads to nonlinear equations.

Fairly recently, thanks to the increasing knowledge of thermal properties and the importance of nonlinear problems in the study of diffusion processes [2, 3], problems of this kind, even in the domain of nuclear technology, are being solved with the application of numerical and analytical methods. Through the use of a method due to Kirchhoff and van Dusen, which involves basically the introduction of a new variables, Pfann [4] solved several one-dimensional problems of heat conduction and one two-dimensional stationary problem.

Biot [5, 6] and Lardner [7] developed several approximate methods for heat conduction problems, based on variational principles. Later on, Hays worked out a variational method, which he first applied to several problems of hydrodynamics [8], and later also to heat conduction problems involving a temperature-dependent thermal conductivity coefficient [9].

Heat conduction in a plate without internal heat generation was studied by Dowty and Howarth [10] using a finite-difference method.

In the first part of the present paper, wherein we study three-dimensional problems, we use the variational theory of Schechter [11], taking into account temperature dependence of the thermophysical properties and studying both stationary and nonstationary conditions. We analyze the plate problem in detail analytically, and then numerically, followed by a short discussion.

2. If we consider the medium to be homogeneous and isotropic (which is close to actuality, for example, in uranium or uranium oxide reactors), the heat conduction differential equation takes the form

$$\operatorname{div}\left[k \operatorname{grad} T(p,\ t)\right] + q(p,\ t) = \rho c\,\frac{\partial T(p,\ t)}{\partial t}, \tag{1}$$

where

$$k = k(T). \tag{2}$$

For the variational method functional analysis is very important, and, in this regard, the theory developed by Mikhlin [12] and Hilbert [13] is of interest.

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We assume that the temperature in a solid of volume V is representable as a sum of two terms: a temperature distribution T^* and a variational distribution δT:

$$T = T^*(p, t) + \delta T(p, t). \tag{3}$$

The thermal conductivity coefficient k, the density ρ, and the volumetric specific heat capacity c may be temperature-dependent. However, for many materials of engineering interest the temperature dependence of ρ and c is negligible. At the same time, the thermal conductivity coefficient k, on the other hand, often has an essential dependence on the temperature

$$k(T) = k(T^* + \delta T) = k^* + \delta k. \tag{4}$$

If on the boundaries of the solid the temperature distribution is known, the thermal flux is equal to zero, and terms in δk may be neglected, then the functional I can be written in the form

$$I = \int_t \int_V \left[\frac{k^*}{2} (\nabla T \times \nabla T) - qT + \rho c T \frac{\partial T^*}{\partial t} \right] dt dV. \tag{5}$$

It is readily seen that Eq. (1) is the Euler–Lagrange equation.

After choosing a base function we can construct the desired solution; however it will contain n undetermined coefficients β_i. The selection of such a base function is very involved. Starting with the Rayleigh–Ritz method, we have

$$\frac{\partial I}{\partial \beta_i} = 0, \quad i = 0, 1, 2, 3, \ldots, n. \tag{6}$$

From n of these equations we determine the n coefficients β_i.

Two-Dimensional Plate

Placing the origin of the x-axis at the center of the plate (the other two dimensions are assumed to be of a higher order), we write the differential Eq. (1) in the form

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + q = \rho c \frac{\partial T}{\partial t} \tag{7}$$

with the boundary and initial conditions

$$\begin{aligned} &T = 0 \quad t > 0, \qquad x = \pm L, \\ &T = T_r \quad t = 0, \quad -L < x < +L. \end{aligned} \tag{8}$$

The operator (5) may be simplified:

$$I = \int_t \int_x \left[\frac{k^*}{2} \left(\frac{\partial T}{\partial x} \right)^2 - qT + \rho c T \frac{\partial T^*}{\partial t} \right] dt dx. \tag{9}$$

We introduce the following dimensionless quantities:

$$\zeta = x/L, \ \tau = a_0 t/L^2, \ \theta = T/T_r; \tag{10}$$

$$q^* = qL^2/k_0 T_r; \tag{11}$$

$$k = k_0 (1 + \sigma\theta). \tag{12}$$

We assume that the thermal conductivity coefficient is a linear function of the temperature and that the remaining parameters are constant. Equations (7)-(9) then transform into the following equations:

$$\frac{\partial}{\partial \zeta}\left(\frac{k}{k_0} \frac{\partial \theta}{\partial \zeta} \right) + q^* = \frac{\partial \theta}{\partial \tau}; \tag{13}$$

$$\begin{aligned} &\theta = 0 \quad \tau > 0, \quad \zeta = \pm 1; \\ &\theta = 1 \quad \tau = 0, \quad -1 < \zeta < +1; \end{aligned} \tag{14}$$

$$I=\int_0^{\infty}\int_{-1}^{+1}\left[\frac{1+\sigma\theta^*}{2}\left(\frac{\partial\theta}{\partial\zeta}\right)^2+\theta\frac{\partial\theta^*}{\partial\tau}-\theta\right]d\tau d\zeta. \tag{15}$$

In order to make clear that the choice of base function is one of the most involved problems of the variational method, we consider two functions, which, although analogous in form, lead to absolutely different results. At first we consider the following basis function:

$$\theta=\sum_m e^{-\beta_m\tau}A_m\cos\lambda_m\zeta+\sum_m B_m\cos\lambda_m\zeta. \tag{16}$$

We see that Eq. (16) satisfies the boundary and initial conditions (14) if

$$\lambda_m=\pi\left(m+\frac{1}{2}\right), \tag{17}$$

$$A_m=\frac{2(-1)^m}{\lambda_m}\left(1-\frac{q^*}{\lambda_m^2}\right), \tag{18}$$

$$B_m=\frac{2q^*(-1)^m}{\lambda_m^3}. \tag{19}$$

We minimize the functional (15) using the basis function (16). We have

$$\frac{\partial I}{\partial\beta_j}=\int_0^{\infty}\int_{-1}^{+1}\left[(1+\sigma\theta^*)\frac{\partial\theta}{\partial\zeta}\cdot\frac{\partial}{\partial\beta_j}\left(\frac{\partial\theta}{\partial\zeta}\right)+\frac{\partial\theta}{\partial\beta_j}\cdot\frac{\partial\theta^*}{\partial\tau}-q^*\frac{\partial\theta}{\partial\beta_j}\right]d\tau d\zeta=0, \tag{20}$$

and, correspondingly, we establish the supplementary condition

$$\theta=\theta^*. \tag{21}$$

Carrying out the integration with

$$N=2n+1,\quad \Gamma=2\gamma+1,\ J=2j+1, \tag{22}$$

where n, γ, j = 0, 1, 2, . . . , m, we obtain a series of nonlinear algebraic expressions in the unknowns β_j

$$\frac{1}{J^2\beta_j}\left[1-\frac{q^*}{\lambda_j^2}-\frac{1}{\beta_j}(\lambda_j^2-q^*)\right]+16\sigma\sum_{\gamma,n}\left\{\left[\left(1-\frac{q^*}{\lambda_\gamma^2}\right)\left(1-\frac{q^*}{\lambda_n^2}\right)\frac{1}{(\beta_j+\beta_n+\beta_\gamma)^2}\right.\right.$$
$$\left.\left.+\frac{4q^*}{\pi^2(\beta_\gamma+\beta_j)^2N^2}\right]+\frac{4q^*}{\pi^2\Gamma^2}\left[\frac{1}{(\beta_n+\beta_j)^2}\left(1-\frac{q^*}{\lambda_n^2}\right)+\frac{4q^*}{\pi^2N^2\beta_j^2}\right]\right\}\frac{(N^2+J^2-\Gamma^2)}{[N^2-(\Gamma+J)^2][N^2-(\Gamma-J)^2]}=0. \tag{23}$$

We use the Newton–Raphson method for solving this system. Putting $q^*=20$, we obtain β_j only up to $\sigma = 0.073$. For large σ the magnitude of the error is of the same order as that of β_0. If we consider Fig. 1a, on which 6 values of β_j are represented (it should be remembered that the ordinary scale must be multiplied by 10 to obtain the β_1-values for various σ-values, by 100 to obtain the values of β_2, β_3, β_4, and by 1000 for the values of β_5) we can then see that while β_2, β_3, β_4, and β_5 stay essentially constant, β_0 and β_1 tend to zero, agreeing with the fact that as the thermal conductivity coefficient increases the value of the temperature at a given instant and at a given location must be less that its value for constant thermal conductivity. Thus the function (16) does not give the right result. Therefore we introduce the new basis function

$$\theta=\sum_n e^{-\beta_n\tau}C_n\cos\lambda_n\zeta+\sum_n D_n\cos\lambda_n\zeta. \tag{24}$$

Relation (24) satisfies the boundary conditions (14) if

$$\lambda_n=\pi\left(n+\frac{1}{2}\right), \tag{17a}$$

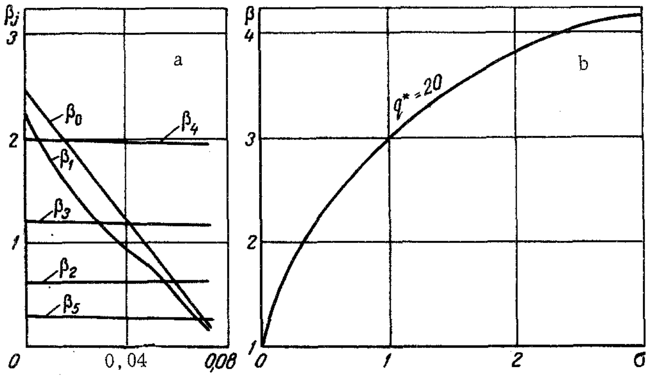

Fig. 1. a) Dependence of β_j on σ; b) dependence of β on σ ($q^* = 20$).

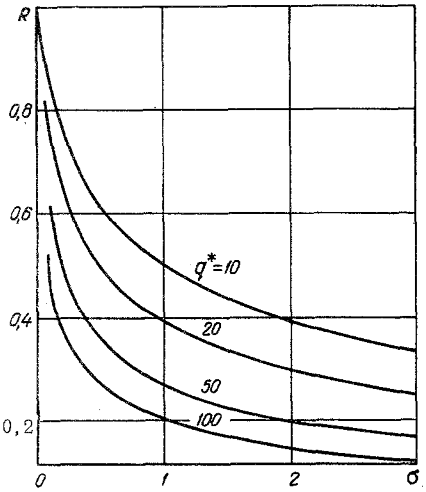

Fig. 2. Dependence of the coefficient R on σ for several values of q^*.

$$D_n = \frac{2Rq^*(-1)^n}{\lambda_n^3}, \tag{25}$$

$$C_n = \frac{2(-1)^n}{\lambda_n}\left(1 - \frac{Rq^*}{\lambda_n^2}\right) \tag{26}$$

and the corresponding value for R is given. This trial function must satisfy all the aforementioned conditions and, in addition, the values of β_h (to distinguish them from the previous values β_j) must be such that for $\sigma > 0$ the temperature for the stationary condition must be reached more quickly than for $\sigma = 0$. We must first determine the value of R. For the stationary condition we choose

$$\theta = \frac{Rq^*}{2}(1 - \zeta^2) \tag{27}$$

as the basis function and, using the same minimization method, we obtain

$$R = \frac{-5 + \sqrt{25 + 20\sigma q^*}}{2\sigma q^*} \tag{28}$$

and graphs of R (Fig. 2) for several values of q^* and σ. Remembering that $R = 1$ for $\sigma = 0$, we obtain, starting from Eq. (27), Fig. 3a, from which we see that for a thermal conductivity coefficient which depends fairly strongly on the temperature, we cannot, as is well known, neglect temperature changes in the stationary case, a fact which is also observable when this same differential equation is solved using the Runge–Kutta method (dashed curve).

A comparison of the two forms of solution shows that the variational method gives plausible results quickly and without difficulties whereas the Runge–Kutta method requires the use of a digital calculator.

We turn now to the calculation of the β_h. Solving Eqs. (20) and (21), using β_h instead of β_j, and substituting θ from Eq. (24) instead of from Eq. (16), we obtain

$$\frac{1}{H^2\beta_h}\left[1 - \frac{Rq^*}{\lambda_h^2} - \frac{1}{\beta_h}(\lambda_h^2 + 3Rq^* - 4q^*)\right] + 16\sigma\sum_{\gamma,n}\left\{\left(1 - \frac{Rq^*}{\lambda_\gamma^2}\right)\left[\left(1 - \frac{Rq^*}{\lambda_n^2}\right)\frac{1}{(\beta_\gamma + \beta_n + \beta_h)^2}\right.\right.$$
$$\left.\left. + \frac{4Rq^*}{\pi^2N^2(\beta_\gamma + \beta_h)^2}\right] + \frac{4Rq^*}{\pi^2\Gamma^2}\left[\frac{1}{(\beta_n + \beta_h)^2}\left(1 - \frac{Rq^*}{\lambda_n^2}\right) + \frac{4Rq^*}{\pi^2N^2\beta_h^2}\right]\right\}\frac{N^2 + H^2 - \Gamma^2}{[N^2 - (\Gamma + H)^2][N^2 - (\Gamma - H)^2]} = 0. \tag{29}$$

For $\sigma = 0$

$$\beta_h = \lambda_h^2 = \frac{\pi^2H^2}{4}, \tag{30}$$

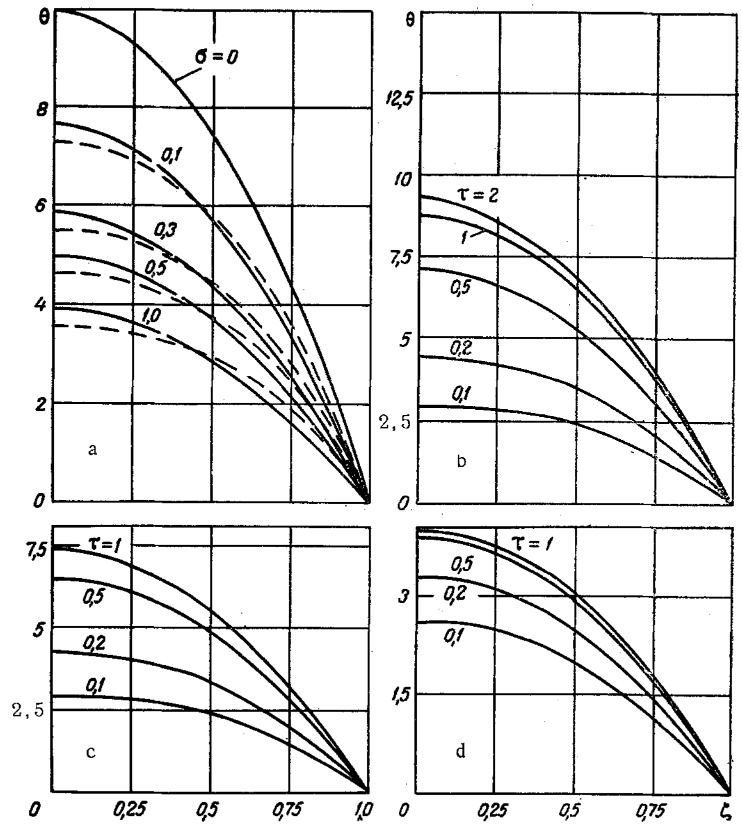

Fig. 3. Dependence of the temperature θ on the dimensionless coordinate ζ: a) stationary problem; b, c, d) nonstationary problem for $\sigma = 0.02$, 0.1, and 1, respectively.

where

$$H = 2h + 1, \; h = 0, 1, 2, \ldots, n. \tag{31}$$

In the form presented the variational method is rather involved and, in order to solve the set of Eqs. (29), completely numerical methods and much machine time is required.

We can, however, consider a trial function with but a single β:

$$\theta = \sum_n e^{-\beta\lambda_n^2\tau} C_n \cos\lambda_n\zeta + \sum_n D_n \cos\lambda_n\zeta, \tag{32}$$

where λ_n, C_n, and D_n are obtained from Eqs. (17a), (25), and (26).

We minimize the functional (15) with respect to β:

$$\frac{dI}{d\beta} = \int_0^\infty \int_{-1}^{+1} \left[(1 + \sigma\theta^*) \frac{\partial\theta}{\partial\zeta} \cdot \frac{\partial}{\partial\beta}\left(\frac{\partial\theta}{\partial\zeta}\right) + \frac{\partial\theta}{\partial\beta} \cdot \frac{\partial\theta^*}{\partial\tau} - q^* \frac{\partial\theta}{\partial\beta} \right] d\tau d\zeta = 0, \tag{33}$$

and, with Eq. (21) in mind, we obtain

$$\sum_h \left(1 - \frac{Rq^*}{\lambda_h^2}\right) \frac{1}{H^2} \left[\frac{16q^*}{\pi^2 H^2} - 1 - \frac{3Rq^*}{\lambda_h^2} + \beta\left(1 - \frac{Rq^*}{\lambda_h^2}\right)\right]$$

$$+ \frac{64\sigma}{\pi^2} \sum_{\gamma,n,h} \left(1 - \frac{Rq^*}{\lambda_h^2}\right) \left\{ H^2 \left(1 - \frac{Rq^*}{\lambda_\gamma^2}\right) \left[\frac{1}{(H^2 + \Gamma^2 + N^2)^2} \left(1 - \frac{Rq^*}{\lambda_n^2}\right) + \frac{4Rq^*}{\pi^2 N^2 (\Gamma^2 + H^2)^2} \right]\right.$$

$$\left. + \frac{4Rq^*}{\pi^2\Gamma^2} \left[\frac{H^2}{(N^2 + H^2)^2} \left(1 - \frac{Rq^*}{\lambda_n^2}\right) + \frac{4Rq^*}{\pi^2 H^2 N^2} \right] \right\} \frac{N^2 + H^2 - \Gamma^2}{[N^2 - (\Gamma - H)^2][N^2 - (\Gamma + H)^2]} = 0. \tag{34}$$

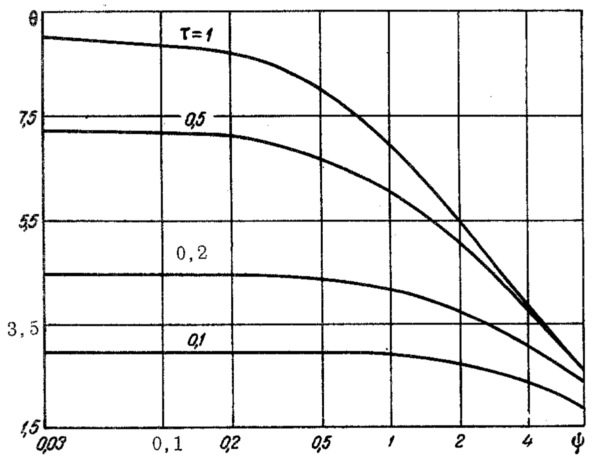

Fig. 4. Dependence of the temperature θ on the ratio ψ for $\xi = 0$.

Solving this equation, we obtain β as a function of σ; the results are shown in Fig. 1b for $q^* = 20$; analogous graphs may be drawn for arbitrary q^*.

Finally, for $\sigma = 0$, $\beta = 1$ and we obtain the temperature distribution obtainable by the usual computational methods.

In Fig. 3b-d we show the temperature distribution θ for several σ for various values of τ.

Finally, in Fig. 4 we show, on a semilog plot, the temperature profile θ as a function of ψ, where

$$\psi = \frac{k - k_0}{k_0}, \tag{35}$$

on the mean plane of the plate ($\xi = 0$) for several time instants τ.

A study of this graph is of interest. At small times ($\tau = 0.1$) the temperature stays almost constant as the thermal conductivity coefficient changes, even when this change exceeds 100%. For times twice as large, noticeable temperature changes are observed when the thermal conductivity coefficient changes by roughly 100%.

A definite temperature stability is also observed at fairly large times ($\tau = 1$) where, in order to obtain noticeable changes of the temperature, it is necessary that ψ change by roughly 10% (we remark that in practice, when $\sigma = 0.1$ and $\tau = 1$, the stationary state has already been attained).

Starting from the considerations detailed above, we see that up to definite values of the parameters we can regard the thermal conductivity as constant or at least use a suitable average value for it. In all remaining cases it is necessary to regard the problem in all its complexity in order to obtain plausible results.

NOTATION

a_0	is the initial thermal diffusivity;
A_m, B_m, C_m, D_n	are the constants from (18), (19), (26), and (25), respectively;
C	is the specific heat capacity;
I	is the functional symbol;
k_0	is the initial thermal conductivity;
k	is the thermal conductivity;
L	is the semithickness of flat plate or semiheight of cylindrical element;
P	is the general point in volume V;
q	is the function of internal heat release per unit time and per unit volume;
q^*	is the dimensionless function of internal heat release determined by Eq. (11);
r, z	are the real and axial coordinates;
R	is the unknown coefficient of sample function;
S	is the total surface at volume V;
T	is the temperature of plate;
T_r	is the initial temperature of flat plate;
t, V	are the time and total volume;
x, y, z	are the orthogonal Cartesian coordinates;
β_m	is the unknown coefficient of sample function (16);
β_n, β'	are the unknown coefficients of sample function (24) and (32);
θ	is the dimensionless temperature determined by Eq. (10);
λ_m, λ_n	are the eigenvalues (17), (17a);
ξ	is the dimensionless orthogonal Cartesian coordinate;
ρ, σ	are the density and angular coefficient (12);
τ	is the dimensionless time, determined by Eq. (10) (Fourier number);
φ	is the angular cylindrical coordinate;
ψ	is the thermal conductivity change to initial thermal conductivity ratio.

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